Non-local Solution of Mixed Integral Equation
in Position and Time with Singular Kernel

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In this paper, we consider the nonlocal Fredholm- Volterra integral equation of the second kind, with discontinuous kernel. The existence and uniqueness of the solution is considered. Then, the Toeplitz matrix method (**TMM**)is used to obtain a nonlocal algebraic system, where the existence and uniqueness solution of the nonlocal algebraic system, can be discussed. The algebraic system is computed numerically, when the historical memory of the problem (nonlocal function) takes three cases: when there is no memory, when the memory is linear and when the memory is nonlinear. Moreover, the estimate error, in each method and each case, is computed. Here, we deduce that, the error in the absence of memory is larger than in the linear memory. Moreover, the error of the nonlinear memory is larger than the linear memory.

Key word: nonlocal Fredholm-Volterra integral equation (nonlocal F-VIE), numerical methods, algebraic system (AS), the error estimate.

1. Introduction:

Many problems in mathematical physics, contact problems in the theory of elasticity and mixed boundary value problems in mathematical physics are transformed into integral equations of linear and nonlinear cases. The books edited by Green [1], Hochstadt [2], Kanwal [3] andSchiavone et al.[4] contained many different methods to solve the linear integral equation analytically. At the same time, the sense of numerical methods takes an important place in solving the linear integral equations. More information for the numerical methods can befound in Linz[5], Golberg [6], Delves and Mohamed[7], Atkinson[8].The **F-VIE** of the first kind in one, two and three dimensions is considered in [9]. In [10-13] the authors consider many numerical methods to solve the integral equations. In all previous work, the nonlocal term (historical memory of the problem) is considered equal zero.

Consider the **nonlocalMIE**of thesecond kind

$$\mu\phi(x,t)=f(x,y)-H(x,t,\phi(x,t))+\lambda\int_{-1}^{1}k(|x-y|)\phi(y,t)dy+\lambda\int_{0}^{t}F(|t-\tau|)\phi(x,\tau)d\tau.\ (1.1)$$

The given two continuous function $f(x,y)$ and $H\left(x,t,\phi(x,t)\right)$ define intheBanachspace

$L_2[-1,1]\times C[0,T]$, $0\le t\le T<1$, where the first function is called the free term and the second

function is known as the memory of the integral equation. The known functions $k\left(|x-y|\right)$ and

$F\left(|t-\tau|\right)$ representthe kernels of Fredholm and Volterra integral terms,respectively. The un-

known function $\phi(x,t)$ represents the solution of (1.1).The constant λ may be complex, has a

physical meaning and μ, defines the kind of integral equation. In this chapter, we will

provethe existence of unique solution of thenonlocalMIE (1.1), under certain conditions, us-

ing Banach fixed-point theorem. Moreover, asuitable quadratic numerical method is used to

reduce the mixed integral equation into **nonlocalSFIEs** of the second kind. Then using the

TMM, as a suitable numerical method to solve the singular integral equation, the **non-

localSFIEs** will reduce to anonlocal algebraic system.Finally,many numerical results are cal-

culated when the kernel takes a logarithmic form and Carleman function formsand the non-

local function takes two forms: linear and nonlinear. Moreover, theestimate error, in each

methodand each case, is computed.

2. The existence and uniqueness of solution

Throughout this section, the existence and uniqueness solution of Eq.(1.1) will be discussed

and proved in the space$L_2[-1,1]\times C[0,T]$,$T<1$,using Banach fixed point theorem. For this, write

Eq.(1.1) in the integra1operator form

$$\mathrm{T}\left(\phi(x,t)\right)=\frac{1}{\mu}[\mathrm{T}_1\left(\phi(x,t)\right)+\lambda\,\mathrm{T}_2\left(\phi(x,t)\right)+\lambda\,\mathrm{T}_3\left(\phi(x,t)\right)], \quad (2.1)$$

$$\mathrm{T}_1\left(\phi(x,t)\right)=-H\left(x,t,\phi(x,t)\right),\ \ \mathrm{T}_2\left(\phi(x,t)\right)=f(x,t)+\int_{-1}^{1}k\left(|x-y|\right)\phi(y,t)dy,\ \ \mathrm{T}_3\left(\phi(x,t)\right)=\int_{0}^{t}F\left(|t-\tau|\right)\phi(x,\tau)d\tau.$$

In addition, we assume the following conditions:

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(i) The given function $H\left(x,t,\phi(x,t)\right)$ with its partial derivatives with respect to position and time is continuous in the space $L_2[\Omega]\times C[0,T]$, for the constant $L > L_1$ and $L > L_2$ satisfies the following conditions:

$(i.a) \left|H\left(x,t,\phi(x,t)\right)\right| \le L_1 \left|\phi(x,t)\right|$ $(i\,b)\left|H\left(x,t,\phi(x,t)\right) - H\left(x,t,\psi(x,t)\right)\right| \le L_2 \left|\phi(x,t) - \psi(x,t)\right|$

Where the norm is defined as $\left\|\phi(x,t)\right\| = \max_{0 \le t \le T}\int_0^t\left[\int_{-1}^1\left|\phi(x,\tau)\right|^2 dx\right]^{1/2} d\tau$.

(ii) The Fredholm kernel satisfies $\left[\int_{-1}^1\int_{-1}^1 k\left(|x-y|\right)^2 dy dx\right] = M^2$,   ($M$ is a constant).

(iii) The discontinuous function $F(|t-\tau|)$ is absolutely integrable with respect to τ for all

$0 \le t \le T < 1$, and satisfies $\int_0^t F(|t-\tau|)d\tau = S$, ($S$ is constant).

(iv)The given function $f(x,t)$ with its partial derivatives with respect to position x and time t are continuous in the space $L_2[-1,1]\times C[0,T]$ and its norm is defined as

$\left\|f(x,t)\right\| = \max_{0 \le t \le T}\left|\int_0^t\left\{\int_{-1}^1 f^2(x,\tau)dx\right\}^{\frac{1}{2}} d\tau\right| = V = $ costant.

Theorem 1: If the conditions (i) − (iv) are satisfied, then Eq. (1.1) has a unique solution

$\phi(x,t)$ in the space $L_2[-1,1]\times C[0,T]$ inside the sphere of radius ρ such that:

$$\rho = V / [\mu - \left(L + |\lambda|M + |\lambda|TS\right)], \quad \left(L + |\lambda|M + |\lambda|TS\right) < \mu. \blacksquare$$

The proof of this theorem depends on the following lemmas.

Lemma 1: Suppose that, the conditions (i-a) and (ii) - (iv) are verified, then the operator T defined by Eq. (2.1) maps the space $L_2[-1,1]\times C[0,T]$ into itself.

Proof: In the light of Eq. (2.1), the normality of the integral operator T after using the conditions (i-a) and (iv) and applying Cauchy Schwarz inequality, yields

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$$|\mu|\,\|\mathrm{T}\phi\| \le V + \mathrm{L}_1\|\phi\| + |\lambda|\left\|\left(\int_{-1}^{1}\left|k\left(|x-y|\right)\right|^2 dy\right)^{\frac{1}{2}}\left(\int_{-1}^{1}\left|\phi(y,t)\right|^2 dy\right)^{\frac{1}{2}}\right\| + |\lambda|\left\|\int_{0}^{t}|F(|t-\tau|)|\,d\tau\int_{0}^{t}|\phi(x,\tau)|\,d\tau\right\|.$$

Using condition (ii)-(iii) and the definition of the norm in the Banach space $L_2[-1,1]\times C[0,T]$, we get

$$\|\mathrm{T}\phi\| \le \frac{V}{|\mu|} + \alpha\|\phi\|, \qquad \alpha = \left(\mathrm{L} + |\lambda|M + |\lambda|TS\right)/|\mu|; \quad \max_{0\le t\le T}\|t\| = T. \tag{2.2}$$

The last inequality shows that, the operator T maps the ball $B_\rho \subset L_2[-1,1]\times C[0,T]$ into itself where $\rho = V / \left(|\mu| - \left(\mathrm{L} + |\lambda|M + |\lambda|TS\right)\right)$. Since $\rho > 0, V > 0$, therefore, we have $\alpha < 1$; also, the inequality (2.2) involves the boundedness of the operator T. ◘

Lemma 2: Suppose that, the conditions (i-b)-(iii) are verified, then the operator T defined by Eq. (4.2.1) is a contraction operator in the space $L_2[-1,1]\times C[0,T]$.

Proof: For the two functions $\phi(x,t)$ and $\psi(x,t)$ in the space $L_2[-1,1]\times C[0,T]$, the formula (2.1) leads to

$$\|\mathrm{T}\phi - \mathrm{T}\psi\| \le \|H(x,t,\phi(x,t)) - H(x,t,\psi(x,t))\| + |\lambda|\left\|\int_{-1}^{1}\left|k\left(|x-y|\right)\right|\left|\phi(y,t)-\psi(y,t)\right|dy\right\| + |\lambda|\left\|\int_{0}^{t}|F(|t-\tau|)|\left|\phi(x,\tau)-\psi(x,\tau)\right|d\tau\right\|. \tag{2.3}$$

After applying Cauchy Schwarz inequality and using conditions (i-b)-(iii), the above inequality becomes

$$\|\mathrm{T}\phi - \mathrm{T}\psi\| \le \alpha\|\phi - \psi\|, \qquad \alpha = \left(\mathrm{L} + |\lambda|M + |\lambda|TS\right)/|\mu|. \tag{2.4}$$

Inequality (2.4) shows that, the operator T is continuous in the space $L_2[-1,1]\times C[0,T]$, and then T is a contraction operator under the condition $\alpha < 1$. ◘

Finally, from the previous lemmas 1,2 we deduce that the operator T is contractive in $L_2[-1,1]\times C[0,T]$. Therefore, from Banach fixed point theorem, T has a unique fixed point $\phi(x,t) \in L_2[-1,1]\times C[0,T]$ which is the unique solution of (1.1) ●

3 System of nonlocal Fredholm integral equations

In this section, a quadratic numerical method is used; seeDelves and Mohamed [7] and Atkinson [8], to obtain **nonlocalSFIEs** of the second kind, where the existence and uniqueness of the integral system are considered. Moreover, the equivalence between the nonlocal **F-VIE** and the nonlocal **SFIEs**is obtained.

For this, we divide the interval $[0,T]$ into s subintervals, by means of the points $0 = t_0 < t_1 < ... < t_m = T$, where $t = t_i, \tau = t_j, i, j = 0,1,2,...,s$, then usingthe quadrature formula. The formula (4.1.1) can be adapted in the following form

$$\mu_i \phi_i(x) = f_i(x) - H_i(x, \phi_i(x)) + \lambda \sum_{j=0}^{i-1} \omega_j F_{i,j} \phi_j(x) + \lambda \int_{-1}^{1} k(|x-y|) \phi_i(y) dy, \qquad i = 0,1,2,...,s. \quad (3.1)$$

Here, we used the following nations:

$$\phi(x,t_i) = \phi_i(x), \ H(x,t_i,\phi(x,t_i)) = H_i(x,\phi_i(x)); \ F(|t_i - t_j|) = F_{i,j}; \ \mu_i = (\mu - \omega_i F_{i,i}), f(x,t_i) = f_i(x);$$

$$\omega_j = \begin{cases} h_j/2 & j=0, \ j=i \\ h_j & 0<j<i. \end{cases}$$

Here, $h_i = \max_{0 \le j \le m} (t_{j+1} - t_j), h_i$ is the step size of integration, and ω_j are the weights,

The value of i and p depend on the number of derivative of $F(|t-\tau|)$ with respect to t for all $\tau \in [0,T]$. Here, we neglect the term of the error of the quadratic numerical method $O\left(h_i^{p+1}\right)$.

The formula (3.1) represents a system of $(s+1)$ **nonlocalFIEs** and its solution depends on the given function $f_i(x)$, the kind of the kernel $k(|x-y|)$ and the degree of the function $H_i(x,\phi_i(x))$.

If in (4.3.1) $\mu = \omega_i F_{i,i}$, we have homogeneous**nonlocalSFIEs**. While, the system is nonhomogeneous if $\mu_i = (\mu - \omega_i F_{i,i}) \ne 0$.

Definition 1: The estimate error $R_{m,i}$, of the quadratic method, isdetermined by the relation

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$$R_{m,i} = \left| \int_0^t F\left(|t_i - t_j|\right) \phi(x, \tau) d\tau - \sum_{j=0}^i \omega_j F_{i,j} \phi_j(x) \right|, \qquad i = 1, 2, \ldots, s. \blacksquare (3.2)$$

Remark1: Consider $\Phi(x) = \{\phi_0(x), \phi_1(x), \ldots, \phi_i(x), \ldots\}$ be the set of all continuousfunctions in

E,where $\phi_i(x) \in L_2[-1,1]$ for all i, and define on E the norm by

$$\|\phi\|_E = \max_i \left[\int_{-1}^1 |\phi_i(x)|^2 dx \right]^{\frac{1}{2}} = \max_i \|\phi_i\|_{L_2[-1,1]} \qquad , \forall i. (3.3)$$

Then, E is aBanach space.◘

● **The existence of a unique solution of the nonlocalSFIEs**

In order to guarantee the existence of a unique solution of (3.1) in the Banach space E , we

assume the following condition in addition to condition (ii) of theorem 1:

(1). $\max_j \sum_{j=0}^{i-1} |\omega_j F_{i,j}| \leq S^*$, (2). $\|f\|_E = \max_i \|f_i\| = V^*, \ \forall i$.

(3). The functions $H_i(x, \phi_i(x))$, for the constants $L^* > L_1^*$ and $L^* > L_2^*$ satisfies the following

conditions:

$(3a). |H_i(x, \phi_i(x))| \leq L_1^* |\phi_i(x)|, (3b). |H_i(x, \phi_i(x)) - H_i(x, \psi_i(x))| \leq L_2^* |\phi_i(x) - \psi_i(x)|.$

Theorem 2(without proof): The **nonlocal SFIEs** (3.1) has a unique solution$\phi_i(x)$ in the

Banach space E under the condition**s:**

$$|\mu^*| \leq \left(L^* + |\lambda| S^* + |\lambda| M\right); \quad \mu^* = \min_i |\mu_i|. \blacksquare (3.4)$$

Also, when $s \to \infty$, then sum $\sum_{j=0}^i \omega_j F_{i,j} \phi_j(x) \to \int_0^t F\left(|t_i - t_j|\right) \phi(x, \tau) d\tau.$ Thus, the solution of the-

nonlocalSFIEs (3.1) becomes the solution of the **nonlocal MIE**(1.1).

Theorem 3(without proof): If the conditions of theorem 2 are satisfiedthen, the sequence of functions $\Phi_s = \left\{ \left(\phi_i \left(x \right) \right)_s \right\}$ of (3.1) convergence uniformly to the solution $\Phi = \left\{ \phi_i \left(x \right) \right\}$ of (1.1) in the Banach space E.

4 The Toeplitz matrix method

Here, we present the**TMM**to obtain the numerical solution of **nonlocalSFIEs**(3.1) of the second kind with singular kernel. The idea of this method is to obtain a system of $(2N+1)$ **nonlocalAS,** where $(2N+1)$ is the number of theusingdiscretepoint.

For this, and in view of chapter zero, Eq. (3.1) yields

$$\mu_i \phi_i \left(x \right) = f_i \left(x \right) + \lambda \sum_{j=0}^{i-1} \omega_j v_{ij} \phi_j \left(x \right) - H_i \left(x, \phi_i \left(x \right) \right) + \lambda \sum_{n=-N}^{N} D_n^{(i)} \left(x \right) \phi_i \left(x \right). \quad (4.1)$$

Here, we used

$$D_n^{(i)} \left(x \right) = \begin{cases} A_N^{(i)} \left(x \right) & , n = -N \\ A_n^{(i)} \left(x \right) + B_{n-1}^{(i)} \left(x \right) & , -N < n < N \\ B_{N-1}^{(i)} \left(x \right) & , n = N. \end{cases}$$

$$A_n^{(i)} \left(x \right) = \frac{1}{h^*} \left[\left(a + h^* \right) I \left(x \right) - J \left(x \right) \right], \qquad B_n^{(i)} \left(x \right) = \frac{1}{h^*} \left[J \left(x \right) - \left(a \right) I \left(x \right) \right].$$

$$I \left(x \right) = \int_a^{a+h^*} k \left(|x - y| \right) dy = A_n^{(i)} \left(x \right) + B_n^{(i)} \left(x \right), \qquad J \left(x \right) = \int_a^{a+h^*} k \left(|x - y| \right) y \, dy = a A_n^{(i)} \left(x \right) + \left(a + h^* \right) B_n^{(i)} \left(x \right).$$

Putting $x = mh^*$, in (4.1)and using the following notations

$$\phi_i \left(mh^* \right) = \phi_{i,m}, \quad H_i \left(mh^*, \phi_i \left(mh^* \right) \right) = H_{i,m} \left(\phi_{i,m} \right), \quad D_n^{(i)} \left(mh^* \right) = D_{mn}^{(i)}, \quad f_i \left(mh^* \right) = f_{im}.$$

We get the following nonlinear algebraic system

$$\mu_i \phi_{i,m} = f_{i,m} - H_{i,m} \left(\phi_{i,m} \right) + \lambda \sum_{j=0}^{i-1} \omega_j v_{ij} \phi_{j,m} + \lambda \sum_{n=-N}^{N} D_{mn}^{(i)} \gamma_{i,n} \left(\phi_{i,n} \right), \quad -N \leq n, m \leq N; \quad 0 \leq i \leq s. \quad (4.2)$$

The matrix $D_{mn}^{(i)}$ *can be written in the following Toeplitz matrix form*

$$D_{mn}^{(i)} = Q_{mn}^{(i)} - P_{mn}^{(i)}; \quad Q_{mn}^{(i)} = A_n^{(i)} \left(mh^* \right) + B_{n-1}^{(i)} \left(mh^* \right), \quad -N \leq n, m \leq N,$$

$$P_{mn}^{(i)} = \begin{cases} B_{-N-1}^{(i)}\left(mh^*\right) & ,n = -N \\ 0 & ,-N < n < N \\ A_N^{(i)}\left(mh^*\right) & ,n = N. \end{cases}$$

$Q_{mn}^{(i)}$ is called the Toeplitz matrix of order $(2N+1)$, and $P_{mn}^{(i)}$ represents a matrix of order $(2N+1)$ whose elements are zeros except the first and the last rows(columns).

Definition 2. The estimate local error $R_{N,n}$ of the **TMM** is determinedby

$$R_{N,n} = \left| \int_{-1}^{1} k\left(|x-y|\right) \phi_i\left(y\right)dy - \sum_{n=-N}^{N} D_{mn}^{(i)} \phi_i\left(nh^*\right) \right| . \tag{4.3}$$

Definition 4.3. The **TMM** is said to be convergent of order r, if and only if for sufficiently large N, there exists a constant $\delta > 0$ independent on N such that

$$\left\| \phi_i\left(x\right) - \left(\phi_i\left(x\right)\right)_N \right\|_\infty \leq \delta N^{-r}. \tag{4.4}$$

●**The existence and uniqueness solution ofthenonlocal AS of TMM**

To prove the existence ofunique solution of the **nonlocalAS** (4.2) in the space ℓ_∞, we consider the following

Lemma 3(without proof), see [37]: If the kernel $k\left(|x-y|\right)$ of Eq. (1.1) satisfies the following condition

A. $\left\| k\left(|x-y|\right) \right\|_{L_2[-1,1]} \leq M^*$, $(M^*$ is constant),

B. $\lim_{x' \to x} \left\| k\left(|x'-y|\right) - k\left(|x-y|\right) \right\|$; $x, x' \in L_2[-1,1]$.

Then, (i). $\sup\limits_{i,N} \sum\limits_{n=-N}^{N} \left| D_{mn}^{(i)} \right|$ exists, (ii) $\lim\limits_{m' \to m} \sup\limits_{i,N} \sum\limits_{n=-N}^{N} \left| D_{m'n}^{(i)} - D_{mn}^{(i)} \right|$.

Consider the following conditions in addition to condition (1)of theorem 2:

(a) $\left\| f \right\|_{\ell_\infty} = \sup\limits_{i,m} \left| f_{i,m} \right| \leq V_1^*$; $\forall i, m$, $(V_1^*$ is constant)

(b) For the constants $L' > L_1'$ and $L' > L_2'$, $H_{i,p}\left(\phi_{i,p}\right)$ satisfies the conditions:

$(b.1)$ $\left| H_{i,p}\left(\phi_{i,p}\right)\right| \leq L_1' \left|\phi_{i,p}\right|$, $(b.2)$ $\left| H_{i,p}\left(\phi_{i,p}\right) - H_{i,p}\left(\psi_{i,p}\right)\right| \leq L_2' \left|\phi_{i,p} - \psi_{i,p}\right|$.

$(c)\sup_{i,N}\sum_{n=-N}^{N}\left|D_{mn}^{(i)}\right| \leq d$, ($d$ is constant).

Theorem 4(without proof): The **nonlocalAS** (4.2) has a unique solution $\phi_{i,m}$ inthe space ℓ_∞

under the condition

$$\left(L' + |\lambda| S^* + |\lambda| d\right) < \mu^*, \qquad \mu^* = \min_i \left|\mu_i\right|. \text{ (4.5)}$$

If $N \to \infty$, the sum $\sum_{n=-N}^{N} D_{mn}^{(i)} \phi_i\left(nh^*\right) \to \int_{-1}^{1} k\left(|x - y|\right)\phi_i\left(y\right)dy$.

Thus, the solution of the**nonlocal AS** (4.2) becomes the solution of the**nonlocal SFIEs** (3.1).

Definition4: Thefollowing relationdetermines estimate total error $R_{s,N} = R_{s,i} + R_{N,n}$ of (4.2).

$$R_{s,N} = \left| \lambda \int_{-1}^{1} k\left(|x - y|\right)\phi(y,t)dy + \lambda\int_{0}^{t} F\left(|t - \tau|\right)\phi(x,\tau)d\tau - \lambda \sum_{n=-N}^{N} D_{mn}^{(i)} \phi_i\left(nh^*\right) - \lambda\sum_{j=0}^{i}\omega_j F_{ij}\phi_j\left(mh^*\right)\right|. \text{(4.6)}$$

When $s, N \to \infty$, the sum

$$\lambda \sum_{n=-N}^{N} D_{mn}^{(i)} \gamma_i\left(nh^*, \phi_i\left(nh^*\right)\right) + \lambda\sum_{j=0}^{i}\omega_j F_{ij}\phi_j\left(x\right) \to \lambda \int_{-1}^{1} k\left(|x - y|\right)\phi(y,t)dy + \lambda\int_{0}^{t} F\left(|t - \tau|\right)\phi(x,\tau)d\tau.$$

Then the solution of the **nonlocal AS** (4.2) becomes the solution ofthe**nonlocal MIE** (1.1).●

Corollary 1: the total error $R_{s,N}$ satisfies $\lim_{s,N\to\infty} R_{s,N} = 0$.

Proof: From the definition of $R_{s,N}$, we have

$$\left| R_{s,N} \right| \leq \left| \phi_i\left(mh^*\right) - \left(\phi_i\left(mh^*\right)\right)_{s,N}\right| + \sum_{j=0}^{i}\left|\omega_j v_{ij}\right|\left|\phi_j\left(mh^*\right) - \left(\phi_j\left(mh^*\right)\right)_{s,N}\right| + \sum_{n=-N}^{N}\left|D_{mn}^{(i)}\right|\left|\phi_i\left(nh^*\right) - \left(\phi_i\left(nh^*\right)\right)_{s,N}\right|.$$

The above inequality, after using condition (c) of theorem 4, yields

$$\left| R_{s,N} \right| \leq \left\| \phi_i\left(mh^*\right) - \left(\phi_i\left(mh^*\right)\right)_{s,N}\right\|_{\ell_\infty} + \left\|\phi_j\left(mh^*\right) - \left(\phi_j\left(mh^*\right)\right)_{s,N}\right\|_{\ell_\infty}\sum_{j=0}^{i}\left|\omega_j v_{ij}\right| + d\left\|\phi_i\left(nh^*\right) - \left(\phi_i\left(nh^*\right)\right)_{s,N}\right\|_{\ell_\infty}.$$

Since $\| \ \|_{\ell_\infty} \leq \| \ \|_{L_2[-1,1]}$, hence for $x = mh^*, y = nh^*$ and condition (1) of theorem 2, we have

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$$\left|R_{s,N}\right| \le \left\|\phi_i(x) - \left(\phi_i(x)\right)_{s,N}\right\|_{L_2[-1,1]} + S^* \left\|\phi_j(x) - \left(\phi_j(x)\right)_{s,N}\right\|_{L_2[-1,1]} + d \left\|\phi_i(y) - \left(\phi_i(y)\right)_{s,N}\right\|_{L_2[-1,1]}.$$

Thus, the above inequality becomes

$$\left|R_{s,N}\right| \le \left(1 + S^* + d\right) \left\|\Phi - \Phi_{s,N}\right\|_{L_2[-1,1]\times C[0,T]}. \quad (4.7)$$

Since, $\left\|\Phi - \Phi_{s,N}\right\|_{L_2[-1,1]\times C[0,T]} \to 0$ when $s,N \to \infty$ in the space $L_2[-1,1] \times C[0,T]$.

Then, $\lim\limits_{s,N\to\infty} \left|R_{s,N}\right| = 0$. Hence, $\lim\limits_{s,N\to\infty} R_{s,N} = 0$. ●

5 Numerical applications

Here, the **TMM** is used to obtain, numerically the solution of the **nonlocal MIE** (1.1) with singular kernel, using maple 12 program. When the singular kernel takes the logarithmic form $k(x,y) = \ln|x-y|$, and Carleman form $k(x,y) = |x-y|^{-\upsilon}, 0 < \upsilon < \frac{1}{2}$, υ is called Poisson ratio and we take $\upsilon = 0.1, 0.25$ and 0.4. The historical function $H(x,t,\phi(x,t))$ takes linear form $t^2\phi(x,t)$ and nonlinear form $\phi^2(x,t)$.

Application : Consider the **nonlocal F-VIE**

$$\mu\phi(x,t) = f(x,t) - H(x,t,\phi(x,t)) + \lambda\int_{-1}^{1} k(x,y)\phi(y,t)dy + \lambda\int_{0}^{t} t\,\tau^2\phi(x,\tau)d\tau, \quad \text{exact solution } \phi(x,t) = x^2 t^2, (4.5.1)$$
$$\left(\mu = 0.5, \ \lambda = 0.01, \ 0 \le t \le T \le 1\right).$$

**Application** A: **When the singular kernel takes the logarithmic form** $k(x,y) = \ln|x-y|$.

Case (A.I): **When the nonlocal term in the linear form in appl.1.**

$$\mu\phi(x,t) = f(x,t) - t^2\phi(x,t) + \lambda\int_{-1}^{1}\ln|x-y|\phi(y,t)dy + \lambda\int_{0}^{t} t\,\tau^2\phi(x,\tau)d\tau, \quad \phi(x,t) = x^2 t^2.$$

	Case A.I: **TMM** when $H(x,t,\phi(x,t)) = t^2\phi(x,t)$, $(\mu = 0.5, \lambda = 0.01, n_t = 4, n_x = 11.)$								
x	$T = 0.0008$			$T = 0.5$			$T = 0.9$		
	ϕ	ϕ^T	E^T	ϕ	ϕ^T	E^T	ϕ	ϕ^T	E^T
-1	6.40000E-07	6.40654E-07	6.54308E-10	2.50000E-01	2.50173E-01	1.72844E-04	8.10000E-01	8.10365E-01	3.64654E-04
-0.8	4.09600E-07	4.08596E-07	1.00384E-09	1.60000E-01	1.59739E-01	2.60815E-04	5.18400E-01	5.17942E-01	4.57866E-04
-0.6	2.30400E-07	2.29951E-07	4.49268E-10	9.00000E-02	8.98829E-02	1.17078E-04	2.91600E-01	2.91397E-01	2.03260E-04
-0.4	1.02400E-07	1.02225E-07	1.74750E-10	4.00000E-02	3.99542E-02	4.58365E-05	1.29600E-01	1.29521E-01	7.93433E-05
-0.2	2.56000E-08	2.55618E-08	3.82065E-11	1.00000E-02	9.98961E-03	1.03853E-05	3.24000E-02	3.23815E-02	1.84634E-05

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0	0.00000E+00	2.55618E-12	2.55618E-12	0.00000E+00	4.94289E-07	4.94289E-07	0.00000E+00	9.81427E-08	9.81427E-08
0.2	2.56000E-08	2.55618E-08	3.82065E-11	1.00000E-02	9.98961E-03	1.03853E-05	3.24000E-02	3.23815E-02	1.84634E-05
0.4	1.02400E-07	1.02225E-07	1.74750E-10	4.00000E-02	3.99542E-02	4.58365E-05	1.29600E-01	1.29521E-01	7.93433E-05
0.6	2.30400E-07	2.29951E-07	4.49268E-10	9.00000E-02	8.98829E-02	1.17078E-04	2.91600E-01	2.91397E-01	2.03260E-04
0.8	4.09600E-07	4.08596E-07	1.00384E-09	1.60000E-01	1.59739E-01	2.60815E-04	5.18400E-01	5.17942E-01	4.57866E-04
1	6.40000E-07	6.40654E-07	6.54308E-10	2.50000E-01	2.50173E-01	1.72844E-04	8.10000E-01	8.10365E-01	3.64654E-04

Table (1)
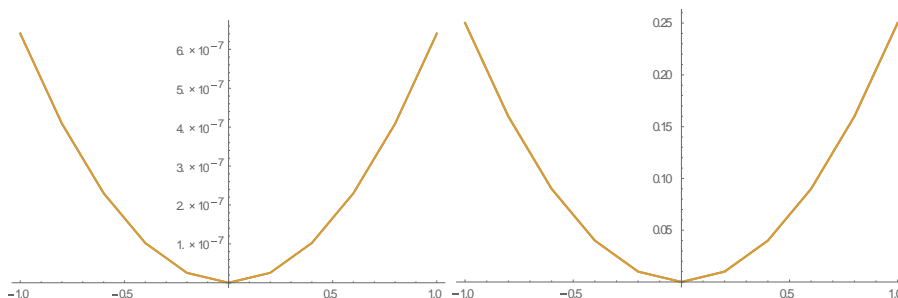
(1-i) $T = 0.0008$ (1-ii) $T = 0.5$
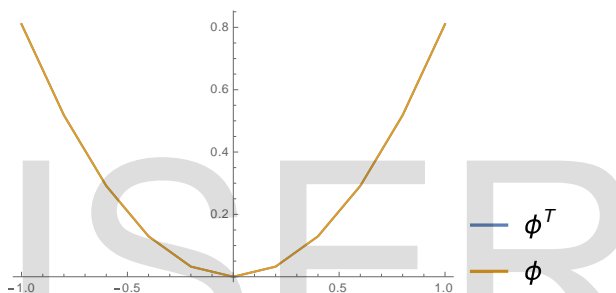
(1-iii) $T = 0.9$

Case A.I :Describes the relation between the exact and numerical solution when $H = t^2\phi(x,t)$ in appl.1,using **TMM** with $(\mu = 0.5, \lambda = 0.01, n_t = 4, n_x = 11)$ at $(T = 0.0008, T = 0.5 \text{ and } T = 0.9)$ in Fig. (1-i), Fig.(1-ii) and Fig. (1-iii), respectively.
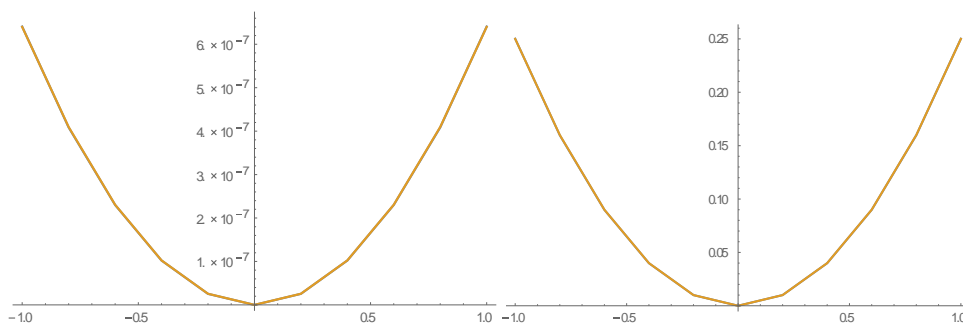
Case (A.II): When the nonlocal term in the nonlinear form in appl.1.

$$\mu\phi(x,t) = f(x,t) - \phi^2(x,t) + \lambda\int_{-1}^{1}\ln|x-y|\phi(y,t)dy + \lambda\int_{0}^{t}t\,\tau^2\phi(x,\tau)d\tau, \quad \phi(x,t) = x^2t^2.$$

	Case A.II: **TMM** when $H(x,t,\phi(x,t)) = \phi^2(x,t)$, $(\mu = 0.5, \lambda = 0.01, n_t = 4, n_x = 11.)$								
x	$T = 0.0008$			$T = 0.5$			$T = 0.9$		
	ϕ	ϕ^T	E^T	ϕ	ϕ^T	E^T	ϕ	ϕ^T	E^T
-1	6.40000E-07	6.40654E-07	6.54307E-10	2.50000E-01	2.50130E-01	1.29695E-04	8.10000E-01	8.10225E-01	2.25334E-04
-0.8	4.09600E-07	4.08596E-07	1.00384E-09	1.60000E-01	1.59761E-01	2.38570E-04	5.18400E-01	5.18010E-01	3.90213E-04
-0.6	2.30400E-07	2.29951E-07	4.49268E-10	9.00000E-02	8.98710E-02	1.29004E-04	2.91600E-01	2.91354E-01	2.45516E-04
-0.4	1.02400E-07	1.02225E-07	1.74750E-10	4.00000E-02	3.99409E-02	5.90548E-05	1.29600E-01	1.29464E-01	1.36262E-04
-0.2	2.56000E-08	2.55618E-08	3.82066E-11	1.00000E-02	9.98522E-03	1.47837E-05	3.24000E-02	3.23580E-02	4.20207E-05
0	0.00000E+00	3.68969E-12	3.68969E-12	0.00000E+00	9.06408E-07	9.06408E-07	0.00000E+00	1.05557E-06	1.05557E-06

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0.2	2.56000E-08	2.55618E-08	3.82066E-11	1.00000E-02	9.98522E-03	1.47837E-05	3.24000E-02	3.23580E-02	4.20207E-05
0.4	1.02400E-07	1.02225E-07	1.74750E-10	4.00000E-02	3.99409E-02	5.90548E-05	1.29600E-01	1.29464E-01	1.36262E-04
0.6	2.30400E-07	2.29951E-07	4.49268E-10	9.00000E-02	8.98710E-02	1.29004E-04	2.91600E-01	2.91354E-01	2.45516E-04
0.8	4.09600E-07	4.08596E-07	1.00384E-09	1.60000E-01	1.59761E-01	2.38570E-04	5.18400E-01	5.18010E-01	3.90213E-04
1	6.40000E-07	6.40654E-07	6.54307E-10	2.50000E-01	2.50130E-01	1.29695E-04	8.10000E-01	8.10225E-01	2.25334E-04

Table (2)

(2-i)$T = 0.0008$ (2-ii)$T = 0.5$
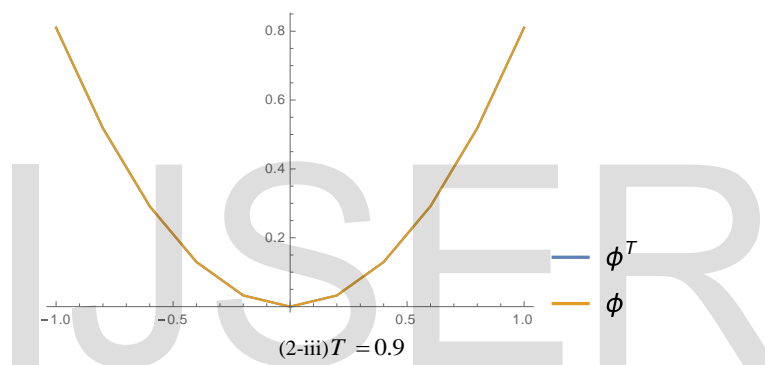
(2-iii)$T = 0.9$

Case A.II: Describes the relation between the exact and numerical solution when $H = \phi^2(x,t)$ in appl.1, using**TMM** with $(\mu = 0.5, \lambda = 0.01, n_t = 4, n_x = 11)$ at $(T = 0.0008, T = 0.5 \text{ and } T = 0.9)$ in Fig. (2-i), Fig.(2-ii)and Fig. (2-iii), respectively.

Application B: When kernel takes the Carleman form $k(x,y) = |x - y|^{-\upsilon}, 0 < \upsilon < \frac{1}{2}$.

Case(B.I):When the nonlocal term in the linear form in appl.1.

$$\mu\phi(x,t) = f(x,t) - t^2\phi(x,t) + \lambda\int_{-1}^{1}|x-y|^{-\upsilon}\phi(y,t)dy + \lambda\int_{0}^{t}t\,\tau^2\phi(x,\tau)d\tau, \quad \phi(x,t) = x^2 t^2.$$

case B.I :**TMM** when $H(x,t,\phi(x,t)) = t^2\phi(x,t)$ $(\mu = 0.5, \lambda = 0.01, n_t = 4, n_x = 11, T = 0.0008)$						
x	ϕ	$\upsilon = 0.1$		$\upsilon = 0.25$		$\upsilon = 0.4$	
		ϕ^T	E^T	ϕ^T	E^T	ϕ^T	E^T
-1	6.40000E-07	6.37062E-07	2.93768E-09	6.36905E-07	3.09515E-09	6.36686E-07	3.31396E-09
-0.8	4.09600E-07	4.12534E-07	2.93368E-09	4.12794E-07	3.19411E-09	4.13166E-07	3.56566E-09
-0.6	2.30400E-07	2.33273E-07	2.87288E-09	2.33406E-07	3.00570E-09	2.33596E-07	3.19600E-09

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| x	ϕ	ϕ^T ($\upsilon=0.1$)	E^T	ϕ^T ($\upsilon=0.25$)	E^T	ϕ^T ($\upsilon=0.4$)	E^T
-0.4	1.02400E-07	1.05243E-07	2.84348E-09	1.05318E-07	2.91843E-09	1.05432E-07	3.03214E-09
-0.2	2.56000E-08	2.84290E-08	2.82905E-09	2.84765E-08	2.87651E-09	2.85552E-08	2.95519E-09
0	0.00000E+00	2.82464E-09	2.82464E-09	2.86384E-09	2.86384E-09	2.93216E-09	2.93216E-09
0.2	2.56000E-08	2.84290E-08	2.82905E-09	2.84765E-08	2.87651E-09	2.85552E-08	2.95519E-09
0.4	1.02400E-07	1.05243E-07	2.84348E-09	1.05318E-07	2.91843E-09	1.05432E-07	3.03214E-09
0.6	2.30400E-07	2.33273E-07	2.87288E-09	2.33406E-07	3.00570E-09	2.33596E-07	3.19600E-09
0.8	4.09600E-07	4.12534E-07	2.93368E-09	4.12794E-07	3.19411E-09	4.13166E-07	3.56566E-09
1	6.40000E-07	6.37062E-07	2.93768E-09	6.36905E-07	3.09515E-09	6.36686E-07	3.31396E-09

Table (3)
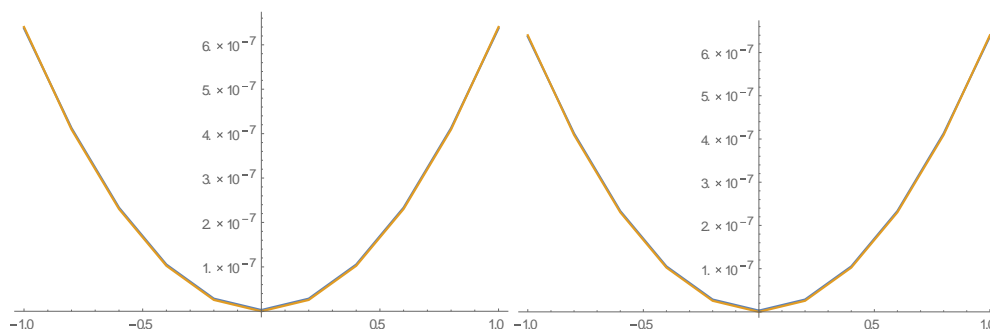
Fig.(3-i)$T = 0.0008, \upsilon = 0.1$ Fig.(3-ii)$T = 0.0008, \upsilon = 0.25$
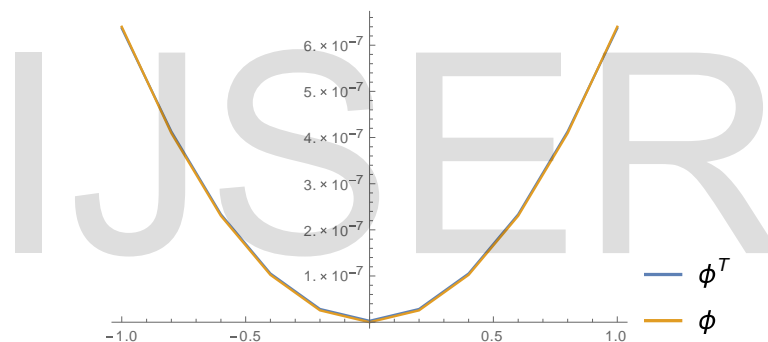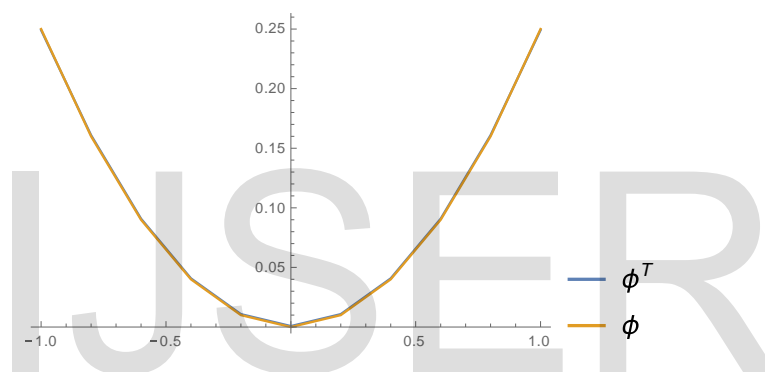

Fig.(3-iii)$T = 0.0008, \upsilon = 0.4$

Case B.I: Describes the relation between the exact and numerical solution, when $H = t^2\phi(x,t)$ in appl.1,using**TMM** with $(\mu = 0.5,\ \lambda = 0.01, T = 0.0008, n_t = 4,\ n_x = 11)$ at $(\upsilon = 0.1, \upsilon = 0.25$ and $\upsilon = 0.4)$ in Fig. (3-i), Fig.(3-ii)and Fig. (3-iii), respectively.

Case B.I.b: **TMM** when $H\left(x,t,\phi(x,t)\right) = t^2\phi(x,t)$ $(\mu = 0.5,\ \lambda = 0.01,\ n_t = 4,\ n_x = 11,\ T = 0.5)$							
x	ϕ	$\upsilon = 0.1$		$\upsilon = 0.25$		$\upsilon = 0.4$	
		ϕ^T	E^T	ϕ^T	E^T	ϕ^T	E^T
-1	2.50000E-01	2.49234E-01	7.66231E-04	2.49193E-01	8.07326E-04	2.49136E-01	8.64203E-04
-0.8	1.60000E-01	1.60758E-01	7.58099E-04	1.60824E-01	8.24323E-04	1.60918E-01	9.18013E-04
-0.6	9.00000E-02	9.07413E-02	7.41252E-04	9.07736E-02	7.73626E-04	9.08193E-02	8.19314E-04
-0.4	4.00000E-02	4.07329E-02	7.32934E-04	4.07500E-02	7.49982E-04	4.07755E-02	7.75508E-04
-0.2	1.00000E-02	1.07288E-02	7.28792E-04	1.07386E-02	7.38556E-04	1.07549E-02	7.54871E-04
0	0.00000E+00	7.27519E-04	7.27519E-04	7.35092E-04	7.35092E-04	7.48683E-04	7.48683E-04

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0.2	1.00000E-02	1.07288E-02	7.28792E-04	1.07386E-02	7.38556E-04	1.07549E-02	7.54871E-04
0.4	4.00000E-02	4.07329E-02	7.32934E-04	4.07500E-02	7.49982E-04	4.07755E-02	7.75508E-04
0.6	9.00000E-02	9.07413E-02	7.41252E-04	9.07736E-02	7.73626E-04	9.08193E-02	8.19314E-04
0.8	1.60000E-01	1.60758E-01	7.58099E-04	1.60824E-01	8.24323E-04	1.60918E-01	9.18013E-04
1	2.50000E-01	2.49234E-01	7.66231E-04	2.49193E-01	8.07326E-04	2.49136E-01	8.64203E-04

Table (4)
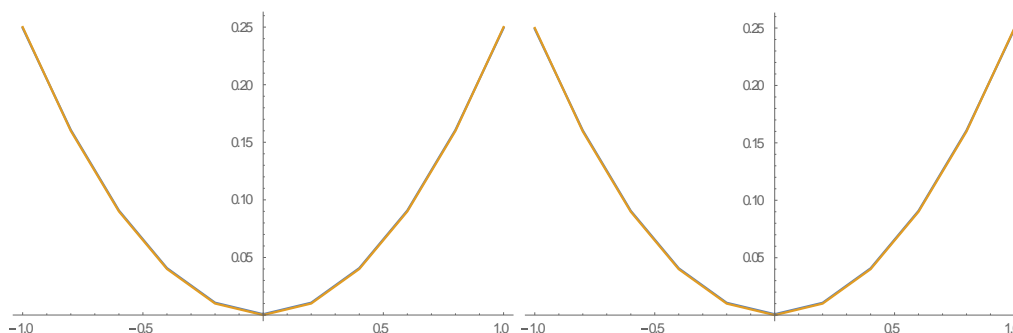
Fig. (4-i)$T = 0.5, \upsilon = 0.1$ Fig.(4-ii)$T = 0.5, \upsilon = 0.25$

Fig.(4-iii)$T = 0.5, \upsilon = 0.4$

Case B.I.b: Describes the relation between the exact and numerical solution when $H = t^2 \phi(x,t)$ in appl.1, using **TMM** with $(\mu = 0.5, \lambda = 0.01, T = 0.5, n_t = 4, n_x = 11)$ at $(\upsilon = 0.1, \upsilon = 0.25 \text{ and } \upsilon = 0.4)$ in Fig. (4-i), Fig.(4-ii) and Fig. (4-iii), respectively.

Case B.I.c: **TMM** when $H(x,t,\phi(x,t)) = t^2 \phi(x,t)$ $(\mu = 0.5, \lambda = 0.01, n_t = 4, n_x = 11, T = 0.9)$						
x	ϕ	$\upsilon = 0.1$		$\upsilon = 0.25$		$\upsilon = 0.4$	
		ϕ^T	E^T	ϕ^T	E^T	ϕ^T	E^T
-1	8.10000E-01	8.08616E-01	1.38417E-03	8.08539E-01	1.46052E-03	8.08434E-01	1.56578E-03
-0.8	5.18400E-01	5.19824E-01	1.42427E-03	5.19945E-01	1.54489E-03	5.20114E-01	1.71434E-03
-0.6	2.91600E-01	2.92980E-01	1.38020E-03	2.93037E-01	1.43699E-03	2.93116E-01	1.51609E-03
-0.4	1.29600E-01	1.30956E-01	1.35571E-03	1.30984E-01	1.38364E-03	1.31025E-01	1.42496E-03
-0.2	3.24000E-02	3.37426E-02	1.34261E-03	3.37568E-02	1.35682E-03	3.37809E-02	1.38090E-03
0	0.00000E+00	1.33845E-03	1.33845E-03	1.34853E-03	1.34853E-03	1.36751E-03	1.36751E-03
0.2	3.24000E-02	3.37426E-02	1.34261E-03	3.37568E-02	1.35682E-03	3.37809E-02	1.38090E-03
0.4	1.29600E-01	1.30956E-01	1.35571E-03	1.30984E-01	1.38364E-03	1.31025E-01	1.42496E-03
0.6	2.91600E-01	2.92980E-01	1.38020E-03	2.93037E-01	1.43699E-03	2.93116E-01	1.51609E-03

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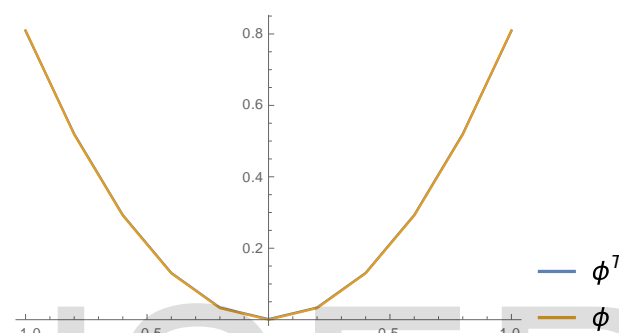| 0.8 | 5.18400E-01 | 5.19824E-01 | 1.42427E-03 | 5.19945E-01 | 1.54489E-03 | 5.20114E-01 | 1.71434E-03 |
| 1 | 8.10000E-01 | 8.08616E-01 | 1.38417E-03 | 8.08539E-01 | 1.46052E-03 | 8.08434E-01 | 1.56578E-03 |

Table (5)
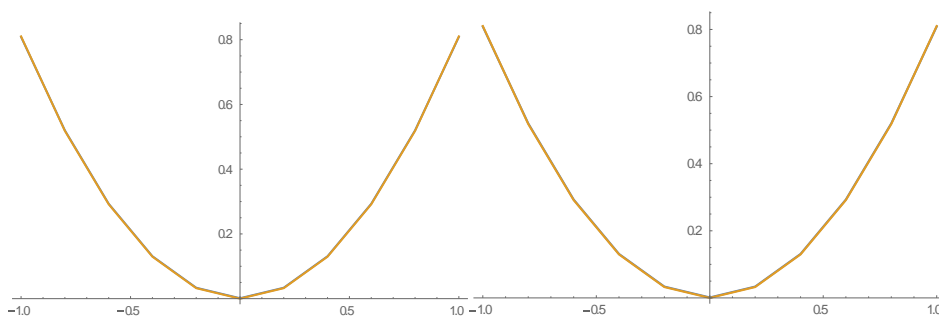
Fig.(5-i) $T = 0.9, \upsilon = 0.1$ Fig.(5-ii) $T = 0.9, \upsilon = 0.25$

$$— \quad \phi^T$$
$$— \quad \phi$$

Fig.(5-iii) $T = 0.9, \upsilon = 0.4$

Case B.I.c: Describes the relation between the exact and numerical solution when $H = t^2\phi(x,t)$ in appl.1,using **TMM** with $(\mu = 0.5, \lambda = 0.01, T = 0.9, n_t = 4, n_x = 11)$ at $(\upsilon = 0.1, \upsilon = 0.25 \text{ and} \upsilon = 0.4)$ in Fig. (5-i), Fig.(5-ii)and Fig. (5-iii), respectively.

Case(B.II): When the nonlocal term in the nonlinear form in appl.1.

$$\mu\phi(x,t) = f(x,t) - \phi^2(x,t) + \lambda\int_{-1}^{1}|x - y|^{-\upsilon}\phi(y,t)dy + \lambda\int_{0}^{t}\tau^2\phi(x,\tau)d\tau, \quad \phi(x,t) = x^2t^2.$$

Case B.II.a: TMM $H(x,t,\phi(x,t)) = \phi^2(x,t)$ when $(\mu = 0.5, \lambda = 0.01, n_t = 4, n_x = 11, T = 0.0008)$						
		$\upsilon = 0.1$		$\upsilon = 0.25$		$\upsilon = 0.4$	
x	ϕ	ϕ^T	E^T	ϕ^T	E^T	ϕ^T	E^T
-1	6.40000E-07	6.37062E-07	2.93768E-09	6.36905E-07	3.09514E-09	6.36686E-07	3.31396E-09
-0.8	4.09600E-07	4.12534E-07	2.93368E-09	4.12794E-07	3.19411E-09	4.13166E-07	3.56566E-09
-0.6	2.30400E-07	2.33273E-07	2.87288E-09	2.33406E-07	3.00570E-09	2.33596E-07	3.19600E-09
-0.4	1.02400E-07	1.05243E-07	2.84348E-09	1.05318E-07	2.91843E-09	1.05432E-07	3.03214E-09
-0.2	2.56000E-08	2.84290E-08	2.82905E-09	2.84765E-08	2.87651E-09	2.85552E-08	2.95519E-09
0	0.00000E+00	2.82465E-09	2.82465E-09	2.86384E-09	2.86384E-09	2.93216E-09	2.93216E-09
0.2	2.56000E-08	2.84290E-08	2.82905E-09	2.84765E-08	2.87651E-09	2.85552E-08	2.95519E-09
0.4	1.02400E-07	1.05243E-07	2.84348E-09	1.05318E-07	2.91843E-09	1.05432E-07	3.03214E-09
0.6	2.30400E-07	2.33273E-07	2.87288E-09	2.33406E-07	3.00570E-09	2.33596E-07	3.19600E-09

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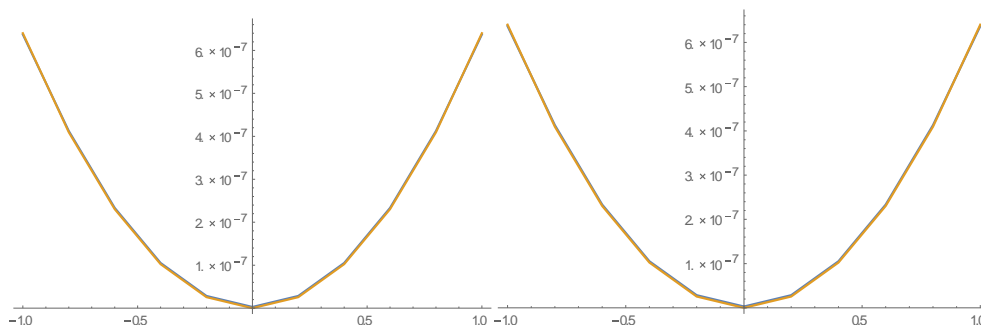| 0.8 | 4.09600E-07 | 4.12534E-07 | 2.93368E-09 | 4.12794E-07 | 3.19411E-09 | 4.13166E-07 | 3.56566E-09 |
| 1 | 6.40000E-07 | 6.37062E-07 | 2.93768E-09 | 6.36905E-07 | 3.09514E-09 | 6.36686E-07 | 3.31396E-09 |

Table(6)

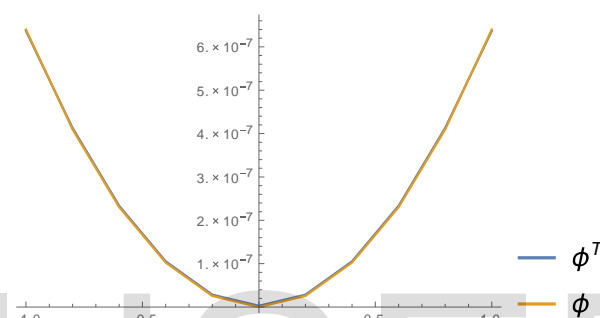Fig.(6-i)$T = 0.0008, \upsilon = 0.1$ Fig.(6-ii)$T = 0.0008, \upsilon = 0.25$

Fig.(6-iii)$T = 0.0008, \upsilon = 0.4$.

Case B.II.a: Describes the relation between the exact and numerical solution, when $H = \phi^2(x,t)$ in appl.1, using **TMM** with $(\mu = 0.5, \lambda = 0.01, T = 0.0008, n_t = 4, n_x = 11)$ at $(\upsilon = 0.1, \upsilon = 0.25$ and $\upsilon = 0.4)$ inFig. (6-i), Fig.(6-ii)and Fig. (6-iii), respectively.

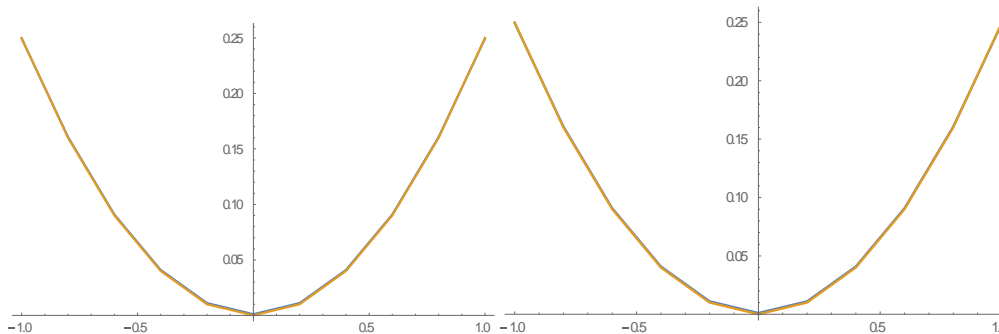| Case B.II.b: **TMM** when $H\left(x,t,\phi(x,t)\right) = \phi^2(x,t)$ ($\mu = 0.5, \lambda = 0.01, n_t = 4, n_x = 11, T = 0.5$)							
x	ϕ	$\upsilon = 0.1$		$\upsilon = 0.25$		$\upsilon = 0.4$	
		ϕ^T	E^T	ϕ^T	E^T	ϕ^T	E^T
-1	2.50000E-01	2.49427E-01	5.73009E-04	2.49396E-01	6.03627E-04	2.49354E-01	6.45882E-04
-0.8	1.60000E-01	1.60698E-01	6.97560E-04	1.60758E-01	7.58407E-04	1.60844E-01	8.44382E-04
-0.6	9.00000E-02	9.08225E-02	8.22542E-04	9.08591E-02	8.59085E-04	9.09108E-02	9.10754E-04
-0.4	4.00000E-02	4.09535E-02	9.53532E-04	4.09773E-02	9.77277E-04	4.10131E-02	1.01306E-03
-0.2	1.00000E-02	1.10575E-02	1.05747E-03	1.10741E-02	1.07407E-03	1.11018E-02	1.10176E-03
0	0.00000E+00	1.09779E-03	1.09779E-03	1.11201E-03	1.11201E-03	1.13715E-03	1.13715E-03
0.2	1.00000E-02	1.10575E-02	1.05747E-03	1.10741E-02	1.07407E-03	1.11018E-02	1.10176E-03
0.4	4.00000E-02	4.09535E-02	9.53532E-04	4.09773E-02	9.77277E-04	4.10131E-02	1.01306E-03
0.6	9.00000E-02	9.08225E-02	8.22542E-04	9.08591E-02	8.59085E-04	9.09108E-02	9.10754E-04
0.8	1.60000E-01	1.60698E-01	6.97560E-04	1.60758E-01	7.58407E-04	1.60844E-01	8.44382E-04
1	2.50000E-01	2.49427E-01	5.73009E-04	2.49396E-01	6.03627E-04	2.49354E-01	6.45882E-04

Table(7)

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Fig.(7-i)$T = 0.5, \upsilon = 0.1$ Fig.(7-ii)$T = 0.5, \upsilon = 0.25$
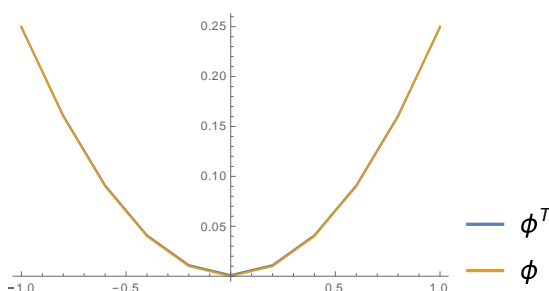

Fig.(7-iii)$T = 0.5, \upsilon = 0.4$

Case B.II.b: Describes the relation between the exact and numerical solution,when $H = \phi^2(x,t)$ in appl.1, using**TMM** with $(\mu = 0.5, \lambda = 0.01, T = 0.5, n_t = 4, n_x = 11.)$ at $(\upsilon = 0.1, \upsilon = 0.25 \text{ and } \upsilon = 0.4)$ in Fig. (7-i), Fig.(7-ii)and Fig. (7-iii), respectively.

| Case B.II.c: **TMM** when $H(x,t,\phi(x,t)) = \phi^2(x,t)$ $(\mu = 0.5, \lambda = 0.01, n_t = 4, n_x = 11, T = 0.9)$							
x	ϕ	$\upsilon = 0.1$		$\upsilon = 0.25$		$\upsilon = 0.4$	
		ϕ^T	E^T	ϕ^T	E^T	ϕ^T	E^T
-1	8.10000E-01	8.09149E-01	8.50915E-04	8.09102E-01	8.97741E-04	8.09038E-01	9.62144E-04
-0.8	5.18400E-01	5.19625E-01	1.22509E-03	5.19729E-01	1.32858E-03	5.19874E-01	1.47377E-03
-0.6	2.91600E-01	2.93285E-01	1.68487E-03	2.93356E-01	1.75598E-03	2.93455E-01	1.85519E-03
-0.4	1.29600E-01	1.31961E-01	2.36053E-03	1.32015E-01	2.41501E-03	1.32096E-01	2.49631E-03
-0.2	3.24000E-02	3.55383E-02	3.13835E-03	3.55833E-02	3.18331E-03	3.56590E-02	3.25895E-03
0	0.00000E+00	3.53065E-03	3.53065E-03	3.57238E-03	3.57238E-03	3.64766E-03	3.64766E-03
0.2	3.24000E-02	3.55383E-02	3.13835E-03	3.55833E-02	3.18331E-03	3.56590E-02	3.25895E-03
0.4	1.29600E-01	1.31961E-01	2.36053E-03	1.32015E-01	2.41501E-03	1.32096E-01	2.49631E-03
0.6	2.91600E-01	2.93285E-01	1.68487E-03	2.93356E-01	1.75598E-03	2.93455E-01	1.85519E-03
0.8	5.18400E-01	5.19625E-01	1.22509E-03	5.19729E-01	1.32858E-03	5.19874E-01	1.47377E-03
1	8.10000E-01	8.09149E-01	8.50915E-04	8.09102E-01	-8.97741E-04	0.80903786	9.62144E-04

Table (8)

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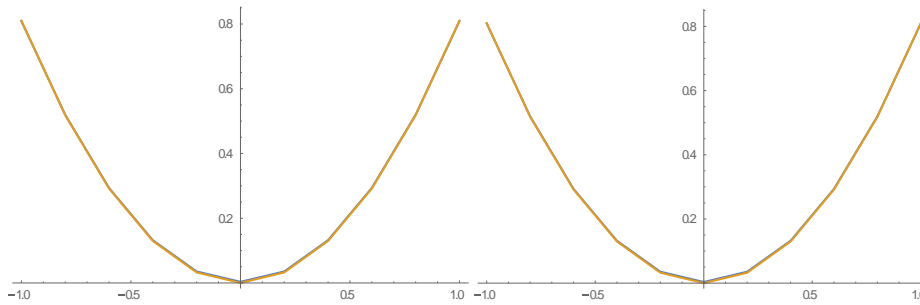
Fig.(8-i)$T = 0.9, \upsilon = 0.1$ Fig.(8-ii)$T = 0.9, \upsilon = 0.25$
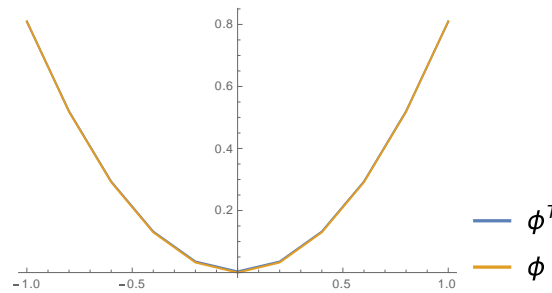

Fig.(8-iii)$T = 0.9, \upsilon = 0.4$

Case B.II.c: Describes the relation between the exact and numerical solution,when $H = \phi^2(x,t)$

in appl.1, using**TMM** with $(\mu = 0.5, \lambda = 0.01, T = 0.9, n_t = 4, n_x = 11.)$ at $(\upsilon = 0.1, \upsilon = 0.25 \text{ and} \upsilon = 0.4)$ and in Fig. (8-

i), Fig.(8-ii)and Fig. (8-iii), respectively.

4.6 Conclusions

2. When the value of thetime increasing The value of theabsolute error is increasing in all cases

of studies.

3. The value of the absolute error is increasing when the value of the Poisson ratio υ is increas-

ing and the time fixed in all cases.

4. The error in the linear non-local function is less than the error in the nonlinear case for all

value of T when x farther from 1,-1.

6. The Max. E. and Min. E. in all cases in the two methods in application 1 and 2are given as

follow:

I- For a logarithmic kernel $k(x,y) = \ln|x - y|$ **and linear nonlocal term** $H(x,t,\phi(x,t)) = t^2\phi(x,t)$ **.**

 I.1- For theToeplitz matrix method,we have:

●in Table (1) for application1,when $(\mu = 0.5)$ and (T=0.0008),Max. E=1.00384E-09 at (x=0.8,-0.8),

Min. E=2.55618E-12 at (x=0). Also,for (T=0.5), Max. E=2.60815E-04 at (x=0.8,-0.8), Min.E

=4.94289E-07 at (x=0). Finally,for(T=0.9)Max. E= 4.57866E-04 at (x=0.8,-0.8), Min. E= 9.81427E-

08 at (x=0).

II- For a logarithmic kernel $k(x,y) = \ln|x - y|$ and nonlinear nonlocal term $H = \phi^2(x,t)$.

II.1- For theToeplitz matrix method,we have:

●in Table (2) for application1,when $(\mu = 0.5)$ and (T=0.0008),Max. E=1.00384E-09 at (x=0.8,-0.8),

Min. E=3.68969E-12 at (x=0). Also,for (T=0.5), Max. E=2.38570E-04 at (x=0.8,-0.8), Min.E =

9.06408E-07 at (x=0). Finally,for(T=0.9)Max. E= 3.90213E-04 at (x=0.8,-0.8), Min. E= 1.05557E-06

at (x=0).

III- For the Carleman kernel $k(x,y) = |x - y|^{-\upsilon}, 0 < \upsilon < \frac{1}{2}$ and linear nonlocal term $H = t^2\phi(x,t)$.

III.1- For theToeplitz matrix method,we have:

●in Table (3) for application1, when $(\mu = 0.5 \text{ and } T = 0.0008)$ with $(\upsilon = 0.1)$,Max. E=2.93768E-09 at

(x=1,-1), Min. E =2.82464E-09 at (x=0). Also,for $(\upsilon = 0.25)$, Max. E=3.19411E-09 at (x=0.8,-0.8),

Min.E = 2.86384E-09   at (x=0). Finally,for $(\upsilon = 0.4)$,Max. E= 3.56566E-09at (x=0.8,-0.8), Min.

E=2.93216E-09 at (x=0).

III.3- For theToeplitz matrix method,we have:

●in Table (4) for application1,when $(\mu = 0.5 \text{ and } T = 0.5)$ with $(\upsilon = 0.1)$,Max. E=7.66231E-04 at

(x=1,-1), Min. E =7.27519E-04 at (x=0). Also,for $(\upsilon = 0.25)$, Max. E=8.24323E-04 at (x=0.8,-0.8),

Min.E = 7.35092E-04   at (x=0). Finally,for $(\upsilon = 0.4)$,Max. E= 9.18013E-04 at (x=0.8,-0.8), Min.

E=7.48683E-04 at (x=0).

III.5- For theToeplitz matrix method,we have:

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●in Table (5) for application1,when $\left(\mu = 0.5 \text{ and } T = 0.9\right)$ with $\left(\upsilon = 0.1\right)$, Max. E=1.42427E-03   at

(x=0.8,-0.8), Min. E =1.33845E-03   at (x=0). Also,for $\left(\upsilon = 0.25\right)$, Max. E=1.54489E-03 at (x=0.8,-0.8),

Min.E = 1.34853E-03at (x=0). Finally,for $\left(\upsilon = 0.4\right)$,Max. E= 9.18013E-04 at (x=0.8,-0.8), Min.

E=7.48683E-04 at (x=0).

IV- For the Carleman kernel $k\left(x,y\right) = \left|x - y\right|^{-\upsilon}, 0 < \upsilon < \frac{1}{2}$ and nonlinear nonlocal term $H = \phi^2\left(x,t\right)$

IV.1- For theToeplitz matrix method,we have:

●in Table (6) for application1,when $\left(\mu = 0.5 \text{ and } T = 0.0008\right)$ with $\left(\upsilon = 0.1\right)$,Max. E=2.93768E-09 at

(x=1,-1), Min. E== 2.82465E-09 at (x=0). Also,for $\left(\upsilon = 0.25\right)$, Max. E=3.19411E-09 at (x=0.8,-0.8),

Min.E =2.86384E-09at (x=0). Finally,for $\left(\upsilon = 0.4\right)$,Max. E= 3.56566E-09 at (x=0.8,-0.8), Min.

E=2.93216E-09 at (x=0).

IV.3- For theToeplitz matrix method,we have:

●in Table (7) for application1,when $\left(\mu = 0.5 \text{ and } T = 0.5\right)$ with $\left(\upsilon = 0.1\right)$,Max. E=1.09779E-03   at

(x=0), Min. E =5.73009E-04 at (x=1,-1). Also,for $\left(\upsilon = 0.25\right)$, Max. E=1.11201E-03 at (x=0), Min.E

=6.03627E-04 at (x=1,-1). Finally,for $\left(\upsilon = 0.4\right)$,Max. E= 1.13715E-03 at (x=0), Min. E= 6.45882E-04

at(x=1,-1).

IV.5- For theToeplitz matrix method, we have:

●in Table (8) for application1,when $\left(\mu = 0.5 \text{ and } T = 0.9\right)$ with $\left(\upsilon = 0.1\right)$,Max. E=3.53065E-03   at

(x=0), Min. E=8.50915E-04 at (x=1,-1). Also,for $\left(\upsilon = 0.25\right)$, Max. E=3.57238E-03 at (x=0), Min.E=

8.97741E-04 at (x=1,-1). Finally,for $\left(\upsilon = 0.4\right)$,Max. E= 3.64766E-03 at (x=0), Min. E = 9.62144E-04

 at (x=1,-1).

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